Intrinsic instability of sonic white holes

U. Leonhardt[1], T. Kiss[1,2,3], and P. Öhberg[1,4]
[1]*School of Physics and Astronomy, University of St Andrews, North Haugh, St Andrews, KY16 9SS, Scotland*
[2]*Research Institute for Solid State Physics and Optics, H-1525 Budapest, P. O. Box 49, Hungary*
[3] *Institute of Physics, University of Pécs, Ifjúság u. 6. H-7624 Pécs, Hungary and*
[4]*Department of Physics, University of Strathclyde, Glasgow G4 0NG, Scotland*

Artificial black holes, such as sonic holes in Bose-Einstein condensates, may give insights into the role of the physics at the event horizon beyond the Planck scale. We show that sonic white holes give rise to a discrete spectrum of instabilities that is insensitive to the analogue of trans-Planckian physics for Bose-Einstein condensates.

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Hawking predicted [1] that black holes generate thermal radiation due to quantum effects [2, 3]. Hawking's effect bridges two vastly different areas of the physical sciences — cosmology and quantum mechanics, but unfortunately, the effect is too feeble to be observable for the known solar-mass or larger black holes. However, condensed-matter or optical analogs may be able to demonstrate the equivalent of Hawking radiation in the laboratory [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In understanding the mechanism of laboratory analogs one could perhaps gain insight into the anatomy of genuine quantum black holes. For example, Hawking's effect appears to rely on a theory that predicts its own demise [17] — radiation outgoing from the event horizon seems to originate from wavelengths beyond the Planck scale where the physics is unknown. Yet the theory of condensed-matter analogues indicates that Hawking's effect is robust against trans-Planckian physics [18]. In this Letter we show that white holes with effective surface gravity α give rise to a discrete spectrum of dynamical instabilities with decay constants

$$\gamma = 2n\alpha\,, \quad n \in \mathbb{Z}\,, \tag{1}$$

whereas black holes may be stable. According to Corley and Jacobson [9], a black-hole white-hole pair should act as a black-hole laser, *i.e.* as an amplifying medium for Hawking radiation with the two horizons forming a resonator, giving rise to a dynamic instability of Hawking radiation. This instability has been attributed to physics beyond the Planck scale. Here we point out that the discrete spectrum of a single white hole is insensitive to the equivalent of trans-Planckian physics within the model we have employed, the sonic hole in a Bose-Einstein condensate. Such a model may be regarded as the *drosophila* of the artificial black holes [10, 12], because it is the simplest system to study theoretically and it is within experimental reach.

Sonic holes are inspiring, because of the mathematical equivalence between the propagation of sound in fluids and of scalar waves in general relativity [5, 6]. Consider sound waves in an irrotational fluid of density ρ_0 and flow $\mathbf{u}$. The velocity potential φ and the density perturbations ρ_s of sound obey the linearized equation of continuity and the linearized Bernoulli equation [19]

$$\partial_t \rho_s + \nabla\cdot(\mathbf{u}\rho_s + \rho_0\nabla\varphi) \;=\; 0\,, \tag{2}$$

$$(\partial_t + \mathbf{u}\cdot\nabla)\varphi + c^2\frac{\rho_s}{\rho_0} \;=\; 0\,. \tag{3}$$

The resulting wave equation in relativistic notation reads

$$D_\nu D^\nu \varphi = \frac{1}{\sqrt{-g}}\,\partial_\mu \sqrt{-g}\, g^{\mu\nu}\partial_\nu\varphi = 0 \tag{4}$$

with the effective space-time metric [6], in d spatial dimensions,

$$g_{\mu\nu} = \left(\frac{\rho_0}{c^3}\right)^{2-2d} \begin{pmatrix} c^2-u^2 & \mathbf{u} \\ \mathbf{u} & -\mathbb{1} \end{pmatrix}. \tag{5}$$

Nonuniform profiles of ρ_0, c^2 and $\mathbf{u}$ may generate effective space-time geometries that are sufficiently rich to possess event horizons [4]. At a sonic horizon the flow exceeds the speed of sound. Sound waves propagating against the current freeze and, in turn, their wavelenghts shrink dramatically. We are interested in effects dominated by this wave catastrophe [16]. In this case we can use the simple one-dimensional model

$$u = -c + \alpha z\,. \tag{6}$$

Here z denotes the spatial coordinate orthogonal to the horizon at $z = 0$, α characterizes the surface gravity or, in our acoustic analog, the gradient of the transonic flow, and ρ_0 and c are assumed to be constant. Strictly speaking, we should complement the flow component (6) in the z direction by appropriate components in the x and y directions, in order to obey the continuity of the flow. But as long as we focus on effects on length scales smaller

than $|c/\alpha|$ we can ignore the other dimensions of the fluid. Equations (2) and (3) have the solutions

$$\varphi = \varphi_0(\tau)\,, \quad \rho_s = \frac{\rho_0}{c\alpha z}\frac{d\varphi}{d\tau}\,, \quad \tau = \frac{\ln(z/z_\infty)}{\alpha} - t\,, \quad (7)$$

with the arbitrary function φ_0 and the constant z_∞, describing wavepackets propagating against the current. Such wavepackets are confined to either $z < 0$ or $z > 0$, depending on the sign of z_∞, which indicates that the place where the flow exceeds the speed of sound, $z = 0$, indeed establishes the acoustic equivalent of the event horizon. Depending on the sign of α, two cases emerge: the sonic black hole and the white hole [10]. The black hole is characterized by a positive velocity gradient α, the flow goes from subsonic to supersonic velocity at the horizon, whereas α is negative for the white hole [10] where the flow slows down from supersonic to subsonic speed. No sound wave can leave the supersonic zone of the black hole and no sound can enter the white hole.

Experimental tests of the subtle quantum effects of the acoustic horizon [4, 20] will take the best superfluids currently available, Bose-Einstein condensates of dilute gases [21, 22]. Moreover, a condensate is one of the simplest physical systems to draw theoretical analogies between the quantum mechanics of fluids and quantum effects of gravity [4]. The mean-field wavefunction ψ_0 of the condensate [22] represents an irrotational fluid,

$$\psi_0 = \sqrt{\rho_0}\, e^{iS_0}\,, \quad \mathbf{u} = \frac{\hbar}{m}\nabla S_0\,, \quad (8)$$

where m denotes the atomic mass. Sound waves are perturbations $\psi - \psi_0$ of the fluid, with

$$\psi = \sqrt{\rho}\, e^{iS}\,, \quad \rho = \rho_0 + \rho_s\,, \quad S = S_0 + \frac{m}{\hbar}\varphi\,. \quad (9)$$

Sound quanta, phonons, are elementary excitations characterized by the Bogoliubov modes u_n and v_n [22, 23], with

$$\psi = \psi_0 + e^{iS_0}(u_n + v_n^*)\,. \quad (10)$$

We determine the modes from the solution (7) of the hydrodynamic equations (2,3) by comparing the representations (9) and (10) in the limit of $|\rho_s/\rho_0| \ll 1$, $|\varphi/S_0| \ll \hbar/m$ [24]. We find the single-frequency Bogoliubov modes of the transonic flow (6),

$$\begin{aligned} u_n &= A_n\left(\frac{\omega}{2\alpha z} + \frac{mc}{\hbar}\right) z^{i\omega/\alpha} e^{-i\omega t}\,, \\ v_n &= A_n\left(\frac{\omega}{2\alpha z} - \frac{mc}{\hbar}\right) z^{i\omega/\alpha} e^{-i\omega t}\,. \end{aligned} \quad (11)$$

Consider the analytic continuation of $z^{i\omega/\alpha}$ to complex z. Suppose that $z^{i\omega/\alpha}$ is analytic on either the upper $(+)$ or lower $(-)$ half plane. Consequently, we get for real and positive z

$$(-z)^{i\omega/\alpha} = e^{-\pi(\pm\omega/\alpha)} z^{i\omega/\alpha}\,. \quad (12)$$

The Bogoliubov modes differ on the two sides of the horizon, depending on the analyticity of $z^{i\omega/\alpha}$, which reflects the independence of the two sides of the sonic horizon.

Transonic flows are notoriously plagued by hydrodynamic instabilities [19], unless they are generated in appropriately designed nozzles such as the Laval nozzle [25] of a rocket engine. Consider unstable elementary excitations corresponding to Bogoliubov modes with complex frequencies ω [24], quasinormal modes [26]. The dynamic equations of the excitations, the Bogoliubov-deGennes equations [22, 23, 24], possess a four-fold symmetry in the complex frequency plane [10, 24]: If ω is the frequency of a solution then there exist solutions for the frequencies ω^*,$-\omega$ and $-\omega^*$ as well. The Hermiticity of the underlying many-body Hamiltonian causes this symmetry [10, 24]. For unstable modes the Bogoliubov scalar products [23, 24] must vanish,

$$\int_{-\infty}^{+\infty} (u_n^* u_{n'} - v_n^* v_{n'})\, dz \;=\; 0\,, \quad (13)$$

$$\int_{-\infty}^{+\infty} (v_n u_{n'} - u_n v_{n'})\, dz \;=\; 0\,, \quad (14)$$

because, in the Bogoliubov-deGennes dynamics [22, 23, 24] the scalar products are stationary [23, 24], whereas the modes (11) are growing or decaying for complex frequencies. Unstable modes in other field theories are subject to similar requirements [26]. As a consequence of the analytic property (12), condition (13) is satisfied in the case of purely imaginary frequencies,

$$\omega = i\gamma\,. \quad (15)$$

Modes (11) with positive γ/α are localized near the horizon at $z = 0$. We deform the contour of integral (14) to a large semicircle with radius r around the origin on, say, the lower half plane, and get

$$\begin{aligned} &\int_{-\infty}^{+\infty} (u_n v_{n'} - v_n u_{n'})\, dz \\ &\quad \sim\; A_n A_{n'} \frac{mc}{\hbar}(\gamma - \gamma') \int_{\pi}^{2\pi} (re^{i\theta})^{-i(\gamma'+\gamma)/\alpha}\, d\theta\,. \end{aligned} \quad (16)$$

The integral vanishes for the decay constants (1). Because of the frequency symmetry of the Bogoliubov-deGennes equations [10, 24] solutions for negative γ/α must exist as well, although we cannot represent them as the acoustic Bogoliubov modes (11), because they would grow in space. Consequently, intrinsic instabilities of sonic horizons, if any, correspond to the discrete spectrum (1). Let us scrutinize the assumptions made.

Close to the horizon the wavelength of sound would shrink beyond all scales and the density (7) would diverge, if the wave equation (4) were universally valid. In the short-wavelength limit we can describe the Bogoli-

ubov modes in the WKB approximation [24, 27],

$$u_n = U_n \exp\left(i\int k\,dz - i\omega t\right), \qquad (17)$$

$$v_n = V_n \exp\left(i\int k\,dz - i\omega t\right). \qquad (18)$$

We obtain the wavenumber k from Bogoliubov's dispersion relation [22] in moving condensates, taking into account the Doppler effect,

$$(\omega - uk)^2 = c^2k^2\left(1+\frac{k^2}{k_c^2}\right), \quad k_c = \frac{mc}{\hbar}. \qquad (19)$$

The group velocity

$$v = \frac{\partial\omega}{\partial k} = u + v', \quad v' = c^2\frac{k}{\omega'}\left(1+\frac{2k^2}{k_c^2}\right), \qquad (20)$$

indicates that the acoustic Compton wavenumber k_c defines the trans-acoustic scale beyond which the excitation velocity in the fluid, v', deviates significantly from the speed of sound. Consider the turning point z_0 where the group velocity (20) vanishes. For single-frequency modes, the excitation flux is conserved [24, 27],

$$\partial_z(U_n^2 - V_n^2)v = 0. \qquad (21)$$

Consequently, the amplitudes U_n and V_n diverge at the turning point. If the trans-acoustic scale k_c were zero the horizon itself would be the turning point. Therefore, we use $|z_0|$ to estimate the spatial range of the trans-acoustic region. For elementary excitations [22, 24],

$$\epsilon = \frac{\hbar\omega}{mc^2} \qquad (22)$$

is a small parameter. We expand z_0 in a power series in $\epsilon^{1/3}$ and solve $v = 0$ to leading order [24],

$$z_0 \sim \frac{c}{\alpha}\frac{3}{2}\sqrt[3]{-1}\left(\frac{\epsilon}{2}\right)^{2/3}. \qquad (23)$$

The trans-acoustic region $|z| \lesssim |z_0|$ is small compared with $|c/\alpha|$. Therefore, neglecting this region in the integral (13) is justified. Consider the solutions of the dispersion relation (19) in the acoustic regime and in the trans-acoustic extreme. We obtain four branches of this fourth-order equation that we characterize by their asymptotics on one side of the horizon. As long as k^2 is much smaller than k_c^2 we get the acoustic asymptotics

$$k \sim \frac{\omega}{k \pm c}, \qquad (24)$$

and in particular,

$$k \sim \frac{\omega}{\alpha z} \qquad (25)$$

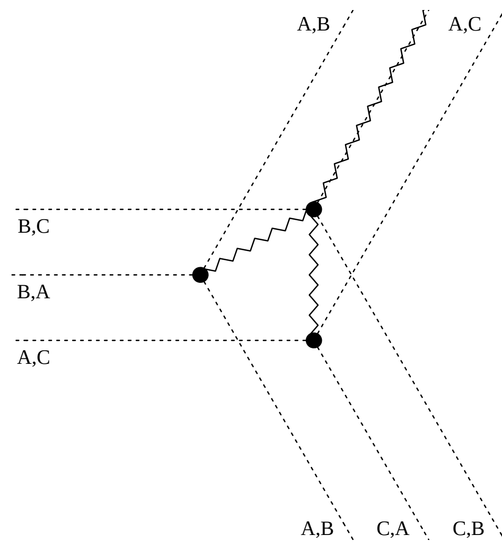

FIG. 1: Stokes lines (dotted lines) of unstable elementary excitations at a sonic white hole. The excitations have purely imaginary frequencies. The Stokes lines originate from the turning points (dots). The jagged lines indicate the branch cuts of the inverted dispersion relation $k = k(\omega)$. Three branches are connected by the branch cuts and may be converted into each other at the Stokes lines, the acoustic branch (A) with asymptotics (25) and two trans-acoustic branches (B) and (C) with asymptotics (26). The pairs of letters indicate which branches of the superposition are potentially converted into each other. The first letter of each pair identifies the exponentially dominant branch (determined numerically). The picture shows that we can construct Bogoliubov modes such that the (B) and (C) branches are not present on the lower half plane, without causing contradictions. Therefore, the unstable elementary excitations of the sonic white hole do indeed obey the asymptotics (25) on the lower half plane.

for sound waves propagating against the current. In the other extreme, when k^2 is much larger than k_c^2, one finds [7]

$$k \sim \pm 2k_c\sqrt{u^2/c^2 - 1} + \frac{\omega u}{c^2 - u^2}. \qquad (26)$$

Our analysis of the instabilities is justified if the Bogoliubov modes obey the asymptotics (25) in one of the complex half planes.

At a turning point z_0 two wavenumber branches of the dispersion relation (19) coincide [20, 24]. Therefore, a solution that starts from a particular k_1 branch on one side of z_0 may become converted into a superposition of the two modes k_1 and k_2 with the common turning point. In other words, turning points may cause scattering. The mode conversion turns out to occur near specific lines in the complex z plane that are called Stokes lines in the mathematical literature [28]. A Stokes line is defined as the line where the difference of the WKB phases, $\int_{z_0}^{z}(k_1 - k_2)\,d\zeta$, is purely imaginary. Each of the three turning points (23) is origin of three Stokes lines, see the Figure. Where the WKB-phase difference is purely imag-

inary, one of the two modes connected by each turning point is exponentially larger than the other. The smaller mode cannot be resolved within the WKB approximation and may gain a component from the larger mode. The single-valuedness of the mode function after a complete circle around the turning points uniquely determines the conversion rules [29]. It follows [29] that the exponentially smaller mode always gains a component from the larger one, if the larger mode is present. Therefore, to avoid unwanted mode conversion the exponentially larger mode should be absent. The Figure shows that the elementary excitations of the sonic white hole can indeed remain on the acoustic branch (25), which justifies the assumptions made to derive the result (1). Note that the localized acoustic excitations with positive γ/α represent decaying modes for the white hole when α is negative. The corresponding growing modes must consist of trans-acoustic excitations with asymptotics (26). However, as a consequence of the four-fold frequency symmetry of the Bogoliubov modes [10, 24], the spectrum (1) is determined by the acoustic modes of the white hole. In contrast, the unstable excitations of sonic black holes, if any, cannot possess the asymptotics (25) on one of the complex half planes. [24]. Otherwise, sonic black holes would be always unstable, and Laval nozzles [12, 25] would be unable to stabilize fluids that turn from subsonic to supersonic speed. Apparently, the opposite process, slowing down supersonic condensates to subsonic speed to form a white hole, is intrinsically unstable, generating breakdown shocks [25].

In conclusion, white holes are best avoided in future experiments to demonstrate Hawking sound in Bose-Einstein condensates. On the other hand, one could still use a toroidal geometry [10], as long as the elementary excitations of the torus do not match the resonances (1). Here the condensate should flow through a constriction where it exceeds the speed of sound, establishing a black-hole horizon followed by a white hole [10]. Because of the periodic boundary condition the spectrum of excitations is restricted. Our theory seems to explain, at least qualitatively, why this toroidal arrangement [10] exhibits instabilities at well-defined lines in the parameter space used. Instead of employing a torus, one could simply push a condensate through the optical equivalent of the Laval nozzle [12, 25] and let the supersonic quantum gas expand into space, like the solar wind [30]. Otherwise, white holes are as unstable as wormholes [31].

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